k-pyramidal 2-factorizations of the complete graph

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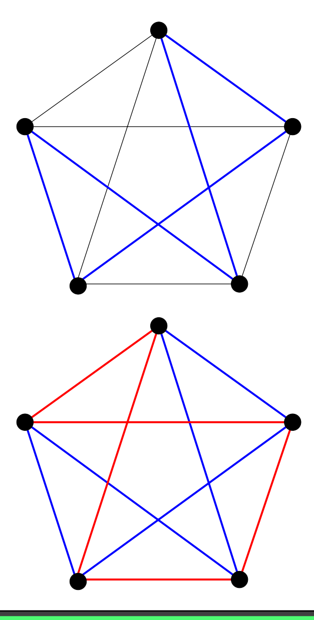
THE MAIN TOPIC

The existence of $2-$factorizations of a complete undirected graph on an odd number of vertices is always guaranteed and it seems clear that the number of non-isomorphic ones becomes huge as the number of vertices increases. Nevertheless, to enumerate them up to isomorphisms is very hard and a general classification seems impossible to achieve without requiring some additional conditions. These conditions may involve the automorphism group of the $2-$factorization as well as the types of $2-$factors. Moreover, it seems reasonable that mostly $2-$factorizations are rigid, i.e., with a trivial full automorphism group. This means the subclass of rigid $2-$factorizations asymptotically covers the class of all $2-$factorizations. In this context, the study of $2-$factorizations with a non trivial automorphism group can lead to either partial or complete classification results. For example the classification of $2-$factorizations which posses an automorphism group having a $2-$transitive action on the vertex-set was achieved in [BBM]. In this last case the request on the group is strong and only few factorizations are permitted in such a particular situation.
Weaker conditions on the automorphism group are requested for the $2-$factorizations presented here. Classification results are not achieved, but many properties are pointed out.

Preliminary definitions and notations

Given a positive integer v we denote by K_v the *complete undirected graph* on v vertices and by $V(K_v)$ and $E(K_v)$ its *vertex-set* and *edge-set* respectively.
A $2-$*factor* of K_v is a $2-$regular spanning subgraph and a $2-$*factorization*, usually denoted by $\mathcal{F}$, is a collection of 2-factors whose edges partition the edge-set of K_v (which necessarily implies v odd).
The number of $2-$factors in a $2-$factorization is $(v-1)/2$ and each 2-factor F of K_v is a set of cycles whose vertices partition $V(K_v)$. By $F(\ell_1, \ldots, \ell_t)$ we denote a 2-factor whose cycles have length $\ell_1, \ldots, \ell_t$, respectively.
A $2-$factor is said to be *Hamiltonian* if it is connected, namely it is formed by a unique cycle. A $2-$factorization is *Hamiltonian* if each $2-$factor is Hamiltonian.
Figures below show a $2-$factor of the complete graph K_5 and a $2-$factorization of K_5. Both factors are Hamiltonian.



k-pyramidal 2-factorizations

Let $\mathcal{F}$ be a $2-$factorization of K_v. An *automorphism group* of $\mathcal{F}$ is a permutation group of the vertex-set which preserves the $2-$factorization.

Definition

A $2-$factorization $\mathcal{F}$ of a complete graph K_v is said to be $k-$*pyramidal* ($k \geq 1$) under the action of a finite group G (or under G for short), if G is an automorphism group of $\mathcal{F}$ which fixes point-wise a subset $X \subset V(K_v)$, with $|X| = k$, and acts regularly, i.e. sharply transitively, on the set $V(K_v) - X$. We also say that G realizes a $k-$pyramidal $2-$factorization.

In this context some natural questions arise:

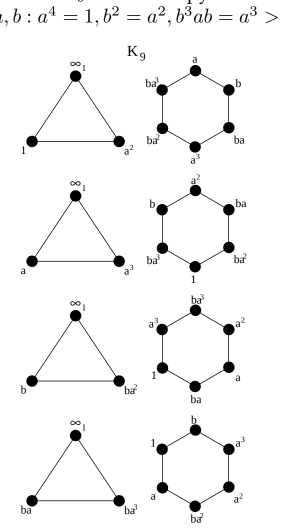
- Which groups can be admissible for such a situation?
- Which kind of $2-$factorizations can be constructed?

The special case $k = 0$ can also be considered. It can be proved that given any group G of odd order, $2-$factorization admitting G as automorphism group with a sharply transitive action on the vertex-set exists. The proof is easy and can be found in [BR1] together with further results.

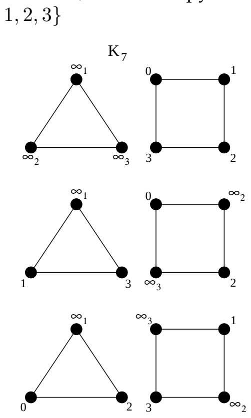
We denote the elements of X by $\infty_1, \infty_2, \ldots, \infty_k$. We identify the elements of $V(K_v) - X$ with G, then the action of G on $V(K_v) - X$ is the product of the group G. This action extends to edges, cycles and factors in the obvious manner.

Some examples

A 2-factorization of K_9 which is 1-pyramidal under $G \cong Q_8 =< a, b : a^4 = 1, b^2 = a^2, b^3ab = a^3 >$



A 2-factorization of K_7 which is 3-pyramidal under $G \cong Z_4 = \{0, 1, 2, 3\}$



The Oberwolfach Problem

The *Oberwolfach problem* (for complete graphs), is the problem of determining the existence of a 2-factorization of K_v in which each 2-factor is isomorphic to a given $F(\ell_1, \ldots, \ell_t)$. Such a problem is denoted by $OP(\ell_1, \ell_2, \ldots, \ell_t)$. If n_i is the number of cycles of length l_i in each 2-factor, the notations $F({}^{n_1}\ell_1, \ldots, {}^{n_r}\ell_r)$ and $OP({}^{n_1}\ell_1, \ldots, {}^{n_r}\ell_r)$ are also used.
The case $OP({}^{r}\ell)$ is completely solved but for most cases $OP(\ell_1, \ell_2, \ldots, \ell_t)$ is not settled. It is believed that apart from $OP(4, 5)$ and $OP(3, 3, 5)$ for which no solutions exist, solutions to $OP(\ell_1, \ldots, \ell_r)$ exist for every other type. You can refer to [A] for a survey.

Main results when $k = 1$

The following results can be found in [BR2] and [BR3].

Proposition 1 *If a $1-$pyramidal $2-$factorization under a group G exists, then G has even order and contains a unique conjugacy class of involutions. The $2-$factorization is Hamiltonian if and only if the group G is symmetrically sequenceable.*

I refer to [K] for the notion of symmetric sequenceable group, or $ss-$group for short.

Proposition 2 *Any 1-pyramidal 2-factorization of a complete graph provides a solution to a suitable Oberwolfach problem and the $2-$factors are in a unique orbit under the action of the group.*

Many new solutions have been constructed. They are listed below. The first column gives Oberwolfach Problems admitting a solution which is 1-pyramidal under the action of $H \times G$, where H is any group of odd order $2w + 1$ and where G is given in the second column. (Q_{2n} denotes the generalized quaternion group of order $2n$, while D_{2n} is the dihedral group of order $2n$).

Solvable OP's	G
$OP(5, {}^{w}8, {}^{2w+1}12)$	Q_{16}
$OP(9, {}^{w}16, {}^{2w+1}24)$	Q_{32}
$OP({}^{3}3, {}^{3w+1}4, {}^{2w}6)$	A_4
$OP(k+1, {}^{w}(2k))$, k even	$ss-$group
$OP(3, {}^{(1+2w)(2t-2)+r}4, {}^{2w+1}6)$	Q_{8t}
$OP(3, {}^{2tw+w+t}4)$	Z_{4t+2}
$OP(3, {}^{2tw+w+t}4)$	D_{4t+2}
$OP(k, {}^{w}(k-1), {}^{2}3, {}^{2w}6)$	$Z_{k+5}, k \geq 7$
$OP(2k+1, {}^{w}(4k), {}^{4w+2}(2k))$	$Z_{6k}, k \geq 2$
$OP({}^{3}(2k+1), {}^{w}(4k), {}^{2w}(4k+2))$	$Z_{6k+2}, k \geq 2$

The dihedral group of order $2n$ can be presented as follows: $D_{2n} =< x, y : y^2 = x^n = 1, xy = yx^{-1} >$

Proposition 3 *There exists a $1-$pyramidal $2-$factorization under a dihedral group of order $2n$ if and only if $2n + 1 = 4t + 3$ for some positive integer t. Every such 2-factorization is of type $(3, 4, ..., 4)$ and it is generated by a 2-factor of the form*

$$F_0 = \{(\infty_1, 1, y)\} \cup \{(x^{a_i}, x^{b_i}, x^{b_i}y, x^{a_i}y) \mid i = 1, ..., t\}$$

where $\{\{a_i, b_i\} \mid i = 1, ..., t\}$ is a starter in Z_{2t+1}. Conversely, every 2-factor of the above form generates a $1-$pyramidal 2-factorization.

Proposition 4 *Let $n = k_1 + k_2 + ... + k_r + 1$ with $k_i \geq 2$ for each i. A 1-pyramidal solution for $OP(3, 2k_1, 2k_2, ..., 2k_r)$ under the action of the cyclic group cannot exist in each of the following cases:*

- $n \equiv 2 \pmod 4$;
- $\frac{n-1}{2} + r$ *is an odd integer.*

Main results when $k > 1$

The following results can be found in [BMR].

First of all the group G is necessarily of even order, say $2n$, and the integer k is odd and between 1 and $2n-1$. The set of cycles with no vertices in $V(K_v) - X$ gives a $2-$factorization of the complete graph on X and any $2-$factorization of K_X can be chosen to be part of a $k-$pyramidal $2-$factorization of K_v.

Proposition 5 *Finite dihedral groups together with abelian and hamiltonian groups of even order realize $k-$pyramidal $2-$factorizations, where k is equal to the number of involutions of the group.*

Proposition 6 *Each group G of order $2n$ realizes a $(2n-1)-$pyramidal $2-$factorization.*

Proposition 7 *Let G be a group of order $2n$. If G realizes a $k-$pyramidal $2-$factorization, then, for each odd t, with $k \leq t \leq 2n-1$, G also realizes a $t-$pyramidal $2-$factorization.*

Given a group G of even order, we denote by λ_G the minimum odd integer k for which G realizes a $k-$pyramidal $2-$factorization. Many examples of groups G with $\lambda_G = 1$ are presented in the side table (case $k = 1$). These examples can be extended to realize $k-$pyramidal $2-$factorizations for each k, with $1 \leq k \leq |G| - 1$.

In what follows we are able to determine λ_G for the class of abelian and dihedral groups.

Proposition 8 *Let G be an abelian group of even order and containing $2t + 1$ involutions. It is $\lambda_G = 2t + 1$.*

Proposition 9 *Let G be a dihedral group of order $2n$. Then*

$$\lambda_G = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 2, \\ \frac{n+4}{2} & \text{if } n \equiv 2 \pmod 4, \\ \frac{n+6}{2} & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

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